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On $\mathbb{Z}_2$-graded polynomial identities of the Grassmann algebra ☆

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Let F be a field of characteristic zero and let E be the Grassmann algebra of an infinite dimensional F-vector space L. In this paper we study the $\mathbb{Z}_2$-graded polynomial identities of E with respect to any fixed $\mathbb{Z}_2$-grading such that L is an homogeneous subspace. We found explicit generators for the ideal, $T_2(E)$, of graded polynomial identities of E and we determine its cocharacter and codimension sequences.

1. Introduction

The Grassmann algebra, E, generated by an infinite dimensional vector space, L, and its $\mathbb{Z}_2$-graded polynomial identities play an important role in Kemer's structure theory on varieties of associative algebras with polynomial identities [10,11]. More precisely, Kemer proved that any associative P.I.-algebra over a field, F, of characteristic zero is P.I.-equivalent to the Grassmann envelope of a finite dimensional associative superalgebra. Moreover the matrix algebras $M_m(E)$ over E and its certain

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subalgebras $M_{p,q}(E)$ $(p+q=m)$, together with the matrix algebras over the ground field, generate the only non trivial prime varieties, as Kemer pointed out in his works.

At the light of this it seems a natural and interesting problem to investigate more closely the structure of the (graded) polynomial identities of these algebras. The complete description of these polynomial identities is known in very few cases only. A classical theorem of Krakowski and Regev gives the codimension sequence and a basis of the polynomial identities of the Grassmann algebra (see [12]). Later, its cocharacter sequence was determined in [13]. Also the structure of the $\mathbb{Z}_2$-graded polynomial identities of E with respect to its natural $\mathbb{Z}_2$-grading is well known, see for instance [8].

In this paper we study the graded polynomial identities of E with respect any $\mathbb{Z}_2$-grading such that the underlying vector space L is an homogeneous subspace. This is equivalent to consider a basis $\mathcal{B}=\{e_1,e_2,\ldots\}$ of L and the $\mathbb{Z}_2$-grading induced on E by any fixed map $\|\cdot\|:\mathcal{B}\to\mathbb{Z}_2$. More precisely if $w=e_{i_1}e_{i_2}\ldots e_{i_k}$ is an element of the basis of E then we define its $\mathbb{Z}_2$-degree by $\|w\|=\|e_{i_1}\|+\cdots+\|e_{i_k}\|$, hence the even component of E with respect this grading is the subspace spanned by the elements w such that $\|w\|=0$ and the odd component is linearly generated by the elements w such that $\|w\|=1$. Clearly the natural $\mathbb{Z}_2$-grading of E comes from the map $|\cdot|:\mathcal{B}\to\mathbb{Z}_2$ given by $|e_i|=1$ for any $e_i\in\mathcal{B}$. On the other hand, given the map $\|\cdot\|$ we can consider the linear automorphism $\varphi:L\to L$ induced by

$$\varphi(e_i):=\begin{cases} e_i & \text{if } \|e_i\|=0,\\ -e_i & \text{if } \|e_i\|=1\end{cases}$$

and we can extend it to an automorphism acting on the whole Grassmann algebra E. So we can consider the $\langle\varphi\rangle$-polynomial identities of E. Notice that, if A is a given algebra and $G=\langle\varphi\rangle$ is the cyclic group generated by an automorphism of A of order 2, then there exists a well known duality between G-identities and $\mathbb{Z}_2$-graded identities, see for example [8]. In this case the even component of A is the eigenspace associated to the eigenvalue 1 and the odd component comes from the eigenspace related to -1. In our case, $L=L_1\oplus L_{-1}$ where the subspaces L_1 and L_{-1} are the eigenspaces of the linear map φ. More precisely, $L_1=\{v\in L\mid\varphi(v)=v\}$ is the subspace spanned from the vectors e_i such that $\|e_i\|=0$ while $L_{-1}=\{v\in L\mid\varphi(v)=-v\}$ is spanned from the vectors e_i such that $\|e_i\|=1$. In [1], Anisimov determined the explicit value of the G-codimension sequences $c_n(E,\varphi)$ associated to the $\langle\varphi\rangle$-polynomial identities of the algebra (E,φ) in the case when the subspace L_1 is infinite dimensional. On the other hand, if $\dim L_1=k$ he proved that $c_n(E,\varphi)=4^{n-\frac{1}{2}}$ if $n\leqslant k$ while

$$2^{n-1}\sum_{t=0}^{k}\binom{n}{t}\leqslant c_n(E,\varphi)\leqslant 2^n\sum_{t=0}^{k}\binom{n}{t}$$

when $k<n$. Very recently, see [15], the second author of this paper found the exact value of $c_n(E,\varphi)$ for the unique open case, by using the algorithm described by Anisimov in his paper.

Here we adopt the language of $\mathbb{Z}_2$-graded polynomial identities. We are able to determine explicitly a generating set for the $\mathbb{Z}_2$-graded polynomial identities of the superalgebra $(E,\|\cdot\|)$. We describe the relations between the space of multilinear graded polynomial identities of $(E,\|\cdot\|)$ and that of ordinary identities of the algebra E. Moreover, we compute the decomposition of the graded cocharacter associated to $(E,\|\cdot\|)$ in its irreducible components and as a consequence we obtain the results about the $\mathbb{Z}_2$-graded codimensions $c_n^{\mathbb{Z}_2}(E,\|\cdot\|)$ in a different way from Anisimov [1] and da Silva [15].

2. Graded cocharacters and codimensions of superalgebras

Let F be a field and let A be a unitary associative F-algebra. We say that A is a *superalgebra*, or a $\mathbb{Z}_2$-*graded algebra*, if $A=A_0\oplus A_1$ where A_0,A_1 are F-subspaces of A satisfying $A_iA_j\subseteq A_{i+j}$ for all $i,j\in\mathbb{Z}_2$. We call A_i the *i-homogeneous component* of A, and we shall write $|a|=i$ to denote the $\mathbb{Z}_2$-homogeneous degree of the homogeneous element $a\in A_i$. A subspace $W\subseteq A$ is homogeneous if and only if $W=(W\cap A_0)\oplus(W\cap A_1)$. If A,B are superalgebras, an algebra homomorphism $\varphi:A\to B$ is a $\mathbb{Z}_2$-*graded homomorphism* if $\varphi(A_i)\subseteq B_i$ for all $i\in\mathbb{Z}_2$.

One defines a free object in the class of superalgebras by considering the free F-algebra over the disjoint union of two countable sets of variables, Y and Z, whose elements are regarded as *even* and *odd* respectively. This means that the $|y_i| = 0$ and $|z_i| = 1$. We shall denote this *free superalgebra* by $F\langle Y \cup Z\rangle$. Its even part is the space spanned by those monomials in which the elements of Z occur in even number. The remaining monomials span the odd component of $F\langle Y \cup Z\rangle$.

A polynomial $f(y_1, \ldots, y_l, z_1, \ldots, z_m)$ in $F\langle Y \cup Z\rangle$ is called a $\mathbb{Z}_2$-graded polynomial identity for a superalgebra A if it is in the kernel of all $\mathbb{Z}_2$-graded homomorphisms $\varphi : F\langle Y \cup Z\rangle \to A$. In other words, f is a graded polynomial identity for A if it vanishes under all possible substitutions of the variables by elements of A with the same parity: the y_i's are replaced by $a_i \in A_0$ and the z_i's by $b_i \in A_1$. One often calls these substitutions *admissible substitutions* for the superalgebra A.

The set $T_2(A)$ of all graded polynomial identities of A is an ideal of the free superalgebra invariant under all graded endomorphisms of $F\langle Y \cup Z\rangle$. It is called the T_2-ideal of (the graded polynomial identities of) A. It is very large in general, and it is more convenient to study the $\mathbb{Z}_2$-*graded multilinear* polynomials lying in it. A natural way of defining $\mathbb{Z}_2$-graded multilinear polynomials is the following:

Definition 1. For $n \in \mathbb{N}$, the vector space

$$V_n^{\mathbb{Z}_2} := \mathrm{span}_F\langle x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)} \mid \sigma \in S_n,\ x_i \in \{y_i, z_i\}\rangle$$

is called the space of $\mathbb{Z}_2$-graded multilinear polynomials of degree n.

Since the characteristic of the ground field F is zero, a standard process of multilinearization shows that $T_2(A)$ is generated, as a T_2-ideal, by the subspaces $V_n^{\mathbb{Z}_2} \cap T_2(A)$. Actually, it is more efficient to study the factor space

$$V_n^{\mathbb{Z}_2}(A) := \frac{V_n^{\mathbb{Z}_2}}{V_n^{\mathbb{Z}_2} \cap T_2(A)}.$$

An effective tool to this end is provided by the representation theory of the symmetric group.

Indeed, one can notice that $V_n^{\mathbb{Z}_2}$ is a S_n-module with respect to the natural left action, and $V_n^{\mathbb{Z}_2} \cap T_2(A)$ is a S_n-submodule, hence the factor space $V_n^{\mathbb{Z}_2}(A)$ is a S_n-module, as well. We shall denote by $\chi_n^{\mathbb{Z}_2}(A)$ its character (the nth $\mathbb{Z}_2$-*graded cocharacter of* A) and by $c_n^{\mathbb{Z}_2}(A)$ its dimension (the nth $\mathbb{Z}_2$-graded codimension of A).

Actually, the study of the structure of $V_n^{\mathbb{Z}_2}(A)$ can be furthermore simplified by considering "smaller" spaces of multilinear polynomials. To be more precise, for fixed l, m, set

$$V_{l,m} := \mathrm{span}_F \left\langle w \text{ monomials of } V_{l+m}^{\mathbb{Z}_2} \,\middle|\, y_1, \ldots, y_l, z_{l+1}, \ldots, z_{l+m} \text{ occur in } w \right\rangle.$$

Setting $n := l + m$, and $S_l \times S_m = Sym(\{1, \ldots, l\}) \times Sym(\{l+1, \ldots, l+m\}) \leqslant S_n$, the space $V_{l,m}$ is a $S_l \times S_m$-module, and the subspace $V_{l,m} \cap T_2(A)$ is a submodule. Therefore one can form the factor $S_l \times S_m$-module

$$V_{l,m}(A) := \frac{V_{l,m}}{V_{l,m} \cap T_2(A)}.$$

We shall denote by $\chi_{l,m}(A)$ its $S_l \times S_m$-character, and by $c_{l,m}(A)$ its dimension.

We briefly recall that if H is a subgroup of a group G and M is an H-module, we can turn M into a G-module by considering the induced G-module structure. In other words, one sets $M^G := F[G] \otimes_{F[H]} M$. This is the so-called G-module induced by M. The relation between the S_n-structure of $V_n^{\mathbb{Z}_2}(A)$ and the $S_l \times S_m$-structure of $V_{l,m}(A)$ is then displayed by the following result (see [2,6]):

Theorem 2. *Let A be a superalgebra. Then for all $n \in \mathbb{N}$*

$$V_n^{\mathbb{Z}_2}(A) \cong \sum_{m=0}^{n} (V_{n-m,m}(A))^{S_n}$$

as S_n-modules. In particular,

$$c_n^{\mathbb{Z}_2}(A) = \sum_{m=0}^{n} \binom{n}{m} c_{n-m,m}(A).$$

In this way the study of the S_n-structure of $V_n^{\mathbb{Z}_2}(A)$ is reduced to the study of the modules $V_{n-m,m}(A)$. We remark that the knowledge of the $S_{n-m} \times S_m$-character of this module is equivalent to the knowledge of the structure of the space $V_n^{\mathbb{Z}_2}(A)$ under the action of the hyperoctahedral group H_n (see [8,9]).

Throughout this paper, with a light abuse of notation, we shall consider $V_{l,m}$ as spanned by the multilinear monomials in the letters $y_1, \ldots, y_l, z_1, \ldots, z_m$. In this case, $S_l \times S_m$ will denote the product of the symmetric groups S_l and S_m acting on $y_1, \ldots, y_l$ and on $z_1, \ldots, z_m$, respectively.

We give a small account on the representation theory of the groups $S_l \times S_m$ $(l + m := n)$. The irreducible $S_l \times S_m$-characters are in one-to-one correspondence with the pairs of partitions (λ, μ) of l and m respectively; in this case we write $\lambda \vdash l$, $\mu \vdash m$, and $|\lambda| = l$, $|\mu| = m$. More precisely, if χ_ν denotes the irreducible $S_{|\nu|}$-character associated to the partition ν, then the irreducible $S_l \times S_m$-character associated to (λ, μ) is $\chi_{\lambda,\mu} = \chi_\lambda \otimes \chi_\mu$.

In order to simplify the notation, we shall often identify the irreducible character χ_ν of the symmetric group with the corresponding partition $\nu = (\nu_1, \ldots, \nu_r)$. So, for instance, we shall write

$$\chi_{l,m}(A) = \sum_{\substack{\lambda \vdash l \\ \mu \vdash m}} m_{\lambda,\mu}\, \lambda \otimes \mu$$

for some multiplicities $m_{\lambda,\mu} = m_{\lambda,\mu}(A) \geqslant 0$.

3. $\mathbb{Z}_2$-gradings on E

Let F be a field of characteristic zero and L be an infinite dimensional vector space over F. Let E be the *Grassmann algebra* generated by L, let $\mathcal{B} = \{e_1, e_2, \ldots\}$ be a linear basis of L and let $\mathcal{E} = \{e_{i_1} e_{i_2} \ldots e_{i_n} \mid n \in \mathbb{N},\ e_{i_1} < e_{i_2} < \cdots < e_{i_n}\}$ be the basis of the Grassmann algebra E generated by L. In this paper we consider the $\mathbb{Z}_2$-gradings of E defined as in the first section. More precisely, let

$$\|\cdot\| : \mathcal{B} \to \mathbb{Z}_2$$

be a fixed map, if $w = e_{i_1} e_{i_2} \ldots e_{i_n} \in \mathcal{E}$ then the set $Supp(w) := \{e_{i_1}, e_{i_2}, \ldots, e_{i_n}\}$ is the *support* of w and we define the $\mathbb{Z}_2$-grading of w by

$$\|e_{i_1} e_{i_2} \ldots e_{i_n}\| = \|e_{i_1}\| + \cdots + \|e_{i_n}\|. \tag{1}$$

When, for all $e_i \in \mathcal{B}$, one has $\|e_i\| = |e_i| = 1 \in \mathbb{Z}_2$ then we obtain the natural $\mathbb{Z}_2$-grading on E, that is:

$$|e_{i_1} e_{i_2} \ldots e_{i_n}| := \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{otherwise.} \end{cases}$$

In this case, let E_0 be the homogeneous component of $\mathbb{Z}_2$-degree 0 and let E_1 be the component of degree 1. It is well known that $E_0 = Z(E)$ is the *center* of E and $ab + ba = 0$ for all $a, b \in E_1$. This means that E satisfies the following graded polynomial identities:

$$[y_1, y_2],\ \ [y_1, z_1],\ \ z_1 z_2 + z_2 z_1.$$

Now, let us consider on E the $\mathbb{Z}_2$-gradings induced by the maps

$$\|\cdot\|_k \quad \|\cdot\|_{k^*} \quad \text{and} \quad \|\cdot\|_\infty$$

defined respectively by:

$$\|e_i\|_k = \begin{cases} 0 & \text{for } i = 1, \ldots, k, \\ 1 & \text{otherwise,} \end{cases} \tag{2}$$

$$\|e_i\|_{k^*} = \begin{cases} 1 & \text{for } i = 1, \ldots, k, \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and

$$\|e_i\|_\infty = \begin{cases} 0 & \text{for } i \text{ even,} \\ 1 & \text{for } i \text{ odd.} \end{cases} \tag{4}$$

Note that, in order to characterize the graded polynomial identities of E and determine its codimension and cocharacter with respect any $\mathbb{Z}_2$-grading $(E, \|\cdot\|)$, it is enough to study them for these gradings.

In our paper, we write

$$(G_k, \|\cdot\|), \quad (G_{k^*}, \|\cdot\|) \quad \text{and} \quad (G_\infty, \|\cdot\|)$$

instead of

$$(E, \|\cdot\|_k), \quad (E, \|\cdot\|_{k^*}) \quad \text{and} \quad (E, \|\cdot\|_\infty),$$

respectively. Moreover we adopt the following notation in order to distingue the elements of the basis of L with respect their $\mathbb{Z}_2$-degree:

- in the superalgebra G_k we put $\eta_i := e_i$ for $i = 1, \ldots, k$ and $\epsilon_i := e_{k+i}$, for all $i = 1, 2, \ldots$;
- similarly, in the superalgebra G_{k^*} we write $\eta_i := e_{k+i}$ for all $i = 1, 2, \ldots$ and $\epsilon_i := e_i$, for $i = 1, \ldots, k$;
- finally, in the superalgebra G_∞ we put $\eta_i := e_{2i}$ and $\epsilon_i := e_{2i-1}$ for all $i = 1, 2, \ldots$

4. Graded PI of G_{k^*} and G_∞

Since in this section also ordinary polynomial identities occur, let us consider the space V_n of multilinear polynomials in the free associative algebra $F\langle X\rangle$. With obvious meaning we can consider the ideal, $T(E)$, of ordinary polynomial identities of the Grassmann algebra E, and the ordinary S_n-modules $V_n(E) = \frac{V_n}{V_n \cap T(E)}$. We shall denote by $\chi_n(E)$ the character of the S_n-module $V_n(E)$, and by $c_n(E)$ its dimension. In this ordinary case, it has been proved in [12,13] that

- $T(E)$ is generated as T-ideal of $F\langle X\rangle$ by the polynomial $[x_1, x_2, x_3]$;
- $c_n(E) = 2^{n-1}$;
- $\chi_n(E) = \sum_{i=0}^{n-1} \nu_i$,

where $\nu_i = (n - i, 1^i) \vdash n$ is the hook partition of n with leg i.

The following is an easy criterion in order to establish if a given multilinear polynomial $f(x_1, \ldots, x_n)$ is a polynomial identity for E.

Lemma 3. *Let $f(x_1, \ldots, x_n) \in V_n$. Then $f \in T(E) \Leftrightarrow$ for any weight map $w : \{1, \ldots, n\} \to \mathbb{Z}_2$ there exist $a_1, \ldots, a_n \in \mathcal{E}$ with pairwise disjoint supports such that, for all i, $|a_i| = w(i) \in \mathbb{Z}_2$ and $f(a_1, \ldots, a_n) = 0$.*

We obtain a similar result for the graded identities of E with respect to its standard $\mathbb{Z}_2$-grading.

Lemma 4. *Let $f(y_1, \ldots, y_l, z_1, \ldots, z_m) \in V_{l,m}$. Then $f \in T_2(E) \Leftrightarrow$ there exist $a_1, \ldots, a_l, b_1, \ldots, b_m \in \mathcal{E}$ with pairwise disjoint supports such that $a_i \in E_0, b_i \in E_1$ and $f(a_1, \ldots, a_l, b_1, \ldots, b_m) = 0$.*

The following result establishes some relations between the $S_l \times S_m$-modules $V_{l,m}(G_\infty)$, $V_{l,m}(G_{k^*})$ and $V_{l+m}(E)$.

Proposition 5. *Let $n = l + m$ and let $\psi_{l,m} : V_n \to V_{l,m}$ be the linear isomorphism induced by the map $x_i \to \begin{cases} y_i & \text{for } i = 1, \ldots, l \\ z_{i-l} & \text{otherwise} \end{cases}$. Then*

(a) *For all $l, m \in \mathbb{N}$*

$$\psi_{l,m}(V_n \cap T(E)) = V_{l,m} \cap T_2(G_\infty)$$

and so $V_n(E)$ and $V_{l,m}(G_\infty)$ are $S_l \times S_m$ isomorphic modules;

(b) *If $m \leqslant k$ then*

$$\psi_{l,m}(V_n \cap T(E)) = V_{l,m} \cap T_2(G_{k^*})$$

and so $V_n(E)$ and $V_{l,m}(G_{k^*})$ are $S_l \times S_m$ isomorphic modules.

Proof. Let $m \leqslant k$ and assume that $\psi_{l,m}(f(x_1, \ldots, x_n)) = f(y_1, \ldots, y_l, z_1, \ldots, z_m)$ is a multilinear graded polynomial identity for the superalgebra G_{k^*}. Let $w : \{1, \ldots, n\} \to \mathbb{Z}_2$ be any fixed map, if $i \leqslant l$ we put

$$a_i = \begin{cases} \eta_{2i-1}\eta_{2i} & \text{if } w(i) = 0, \\ \eta_{2i} & \text{otherwise,} \end{cases}$$

while, for $i = l + 1, \ldots, n$, we consider the following elements in G_{k^*}:

$$b_{i-l} = \begin{cases} \varepsilon_{i-l}\eta_{2i-1} & \text{if } w(i) = 0, \\ \varepsilon_{i-l} & \text{otherwise.} \end{cases}$$

Then $f(a_1, \ldots, a_l, b_1, \ldots, b_m) = 0$ since $f(y_1, \ldots, y_l, z_1, \ldots, z_m) \in T_2\,(G_{k^*})$, and so, by Lemma 3, $f(x_1, \ldots, x_n)$ is a polynomial identity for the Grassmann algebra E, proving the second point of our statement. Similarly we obtain a proof for the first point. □

As a consequence we have:

Corollary 6. *The graded codimension sequences of the superalgebras G_∞ and G_{k^*} are*:

(a) $c_n^{\mathbb{Z}_2}(G_\infty) = 4^{n-\frac{1}{2}}$
(b) $c_n^{\mathbb{Z}_2}\,(G_{k^*}) = 2^{n-1}\sum_{t=0}^{k}\binom{n}{t}$.

Proof. By Theorem 2, one has $c_n^{\mathbb{Z}_2}(G_\infty) = \sum_{m=0}^{n}\binom{n}{m}c_{n-m,m}(G_\infty)$. By previous lemma $c_{n-m,m}(G_\infty) = c_n(E)$ for all $m \leqslant n$ and so the first point is an immediate consequence of the value of $c_n(E) = 2^{n-1}$ given in [12]. Let us consider now the superalgebra G_{k^*}. In this case the polynomial $z_1 \cdots z_{k+1}$ is a graded identity of G_{k^*}. Hence, $c_{n-m,m}\,(G_{k^*}) = 0$ for all $m \geqslant k + 1$ and so, by the previous lemma, we obtain: $c_n^{\mathbb{Z}_2}\,(G_{k^*}) = \sum_{m=0}^{n}\binom{n}{m}c_{n-m,m}\,(G_{k^*}) = \sum_{m=0}^{\min\{k,n\}}\binom{n}{m}2^{n-1} = 2^{n-1}\sum_{t=0}^{k}\binom{n}{t}$ since $\binom{n}{t} = 0$ for $t > n$. □

We also have the following description of the cocharacter sequences.

Corollary 7. *Let $\lambda_s = (l - s, 1^s) \vdash l$ and $\mu_t = (1 + t, 1^{m-t-1}) \vdash m$ be the hook partitions of l and m with leg s and arm t respectively. Let $k \in \mathbb{N}$, then the graded cocharacter sequences of the superalgebras G_∞ and G_{k^*} are the following*:

(a) • $\chi_{l,0}(G_\infty) = \sum_{s=0}^{l-1}\lambda_s \otimes \emptyset\ (l \geqslant 1)$;
• $\chi_{0,m}(G_\infty) = \sum_{t=0}^{m-1}\emptyset \otimes \mu_t\ (m \geqslant 1)$;
• $\chi_{l,m}(G_\infty) = \sum_{s=0}^{l-1}\sum_{t=0}^{m-1}2\,(\lambda_s \otimes \mu_t)\ \ (l, m \geqslant 1)$;
(b) • $\chi_{l,0}\,(G_{k^*}) = \sum_{s=0}^{l-1}\lambda_s \otimes \emptyset\ (l \geqslant 1)$;
• $\chi_{0,m}\,(G_{k^*}) = \sum_{t=0}^{m-1}\emptyset \otimes \mu_t\ (1 \leqslant m \leqslant k)$;
• $\chi_{l,m}\,(G_{k^*}) = \sum_{s=0}^{l-1}\sum_{t=0}^{m-1}2\,(\lambda_s \otimes \mu_t)\ \ (l \geqslant 1,\ 1 \leqslant m \leqslant k)$;
• $\chi_{l,m}\,(G_{k^*}) = 0\ (l \geqslant 0,\ \ m \geqslant k + 1)$.

Proof. Let $n = m + l$, if $m \geqslant k + 1$ then, as we said above, $V_{l,m}\,(G_{k^*}) = 0$. In the other cases, by Proposition 5, the spaces $V_n(E)$, $V_{l,m}(G_\infty)$ and $V_{l,m}\,(G_{k^*})$ are $S_l \times S_m$-isomorphic modules. Hence the result follows from a straightforward computation using the decomposition of $\chi_n(E) = \sum_{i=0}^{n-1}\left(n - i, 1^i\right)$ given in [13] and the representation theory of the symmetric group. More precisely, it follows by Branching Rule (see Section 2.8 of [14]) that when we restrict the irreducible representation $\nu_i = \left(n - i, 1^i\right)$ of S_n to its subgroup $S_l \times S_m$ then its $S_l \times S_m$-irreducible components are $\lambda_s \otimes \mu_t$ for some $\lambda_s = (l - s, 1^s) \vdash l$ and $\mu_t = \left(1 + t, 1^{m-t-1}\right) \vdash m$. By Frobenius Reciprocity Law the multiplicity of $\lambda_s \otimes \mu_t$ in the previous decomposition equals the multiplicity $c_{s,t}^i = c_{\lambda_s\mu_t}^{\nu_i}$ of ν_i in the induced representation $(\lambda_s \otimes \mu_t)^{S_n}$. By the Littlewood-Richardson rule (see Section 4.9 of [14]), $c^i_{s,t}$ is the number of semistandard tableau T such that T has shape ν_i/λ_s, content μ_t and moreover the row word of T, π_T, is a reverse lattice permutation. Since ν_i and λ_s are both hook partitions then the skew shape ν_i/λ_s has at most 2 connected components. The first one is a row of length $n-i-(l-s)=m-(i-s)$, the second is a column of height $i-s$. By the previous conditions on the semistandard tableau T we obtain that entries in the row are all 1 while the entries in the column constitute a standard tableau T'. If 1 does not appear in T' then $1+t=m-(i-s)$, on the other hand if one (the first) entry of T' is 1 then $t=m-(i-s)$. Therefore $c^i_{s,t}$ is non zero if and only if either $i-s+t=m-1$ or $i-s+t=m$, in both cases one has $c^i_{s,t}=1$ since the semistandard tableau T is uniquely determined. As a consequence we obtain the desired conclusion on the decomposition of $\chi_{l,m}(E,\|\cdot\|)$. More precisely, its irreducible component $\lambda_s\otimes\mu_t$ comes from the irreducible components $\nu_{s+m-1-t}$ and ν_{s+m-t} of $\chi_n(E)$. $\square$

Let us recall the following remarkable fact:

Remark 8. Since the polynomial $[x_1,x_2,x_3]$ is an ordinary polynomial identity of the Grassmann algebra E then $T_2(E,\|\cdot\|)$ contains the polynomials $[u_1,u_2,u_3]$ for all $u_i\in Y\cup Z$. Moreover, as in the proof of Lemma 1.4.2 of [4] it follows that the polynomials $[u_1,u_2][u_3,u_4]+[u_1,u_3][u_2,u_4]$ are graded polynomial identities of $(E,\|\cdot\|)$ for any choice of u_i in $\{y_i,z_i\}$, $(i=1,\dots,4)$.

Let us consider the T_2-ideal generated by the previous polynomials. We give:

Definition 9. Let I be the T_2-ideal of $F\langle Y\cup Z\rangle$ generated by the polynomials $[u_1,u_2,u_3]$ for any choice of u_i in $\{y_i,z_i\}$, $(i=1,\dots,3)$.

By the previous remark we have $I\subseteq T_2(E,\|\cdot\|)$, moreover if $\sigma\in S_n$ then

$$[u_{\sigma(1)},u_{\sigma(2)}]\cdots[u_{\sigma(n-1)},u_{\sigma(n)}]=(-1)^\sigma[u_1,u_2]\cdots[u_{n-1},u_n]\ (\mathrm{mod}\ I), \tag{5}$$

where $(-1)^\sigma$ is the signum of the permutation σ.

Now we are able to find the generators for the graded polynomial identities of our superalgebra $(E,\|\cdot\|)$ when the eigenspace $L_1=\{v\in L\mid \|v\|=0\}$ is infinite dimensional. As we said above it is enough to find them for the superalgebras $(E,\|\cdot\|_\infty)=(G_\infty,\|\cdot\|)$ and $(E,\|\cdot\|_{k^*})=(G_{k^*},\|\cdot\|)$. We have:

Theorem 10. *Let $T_2(G_d)$ be the T_2-ideal of the graded polynomial identities for the superalgebra $(G_d,\|\cdot\|)$, $d=\infty,k^*$. Put $U=Y\cup Z$, then*

(a) *$T_2(G_\infty)$ is generated by the set of the following polynomials*:
- $[u_1,u_2,u_3]$

(b) *$T_2\,(G_{k^*})$ is generated by the set of the following polynomials*:
- $[u_1,u_2,u_3]$
- $z_1z_2\cdots z_{k+1}$.

Proof. Let I_k be the T_2-ideal of $F\langle Y\cup Z\rangle$ generated by I together with the monomial $z_1z_2\cdots z_{k+1}$. Clearly $I_k\subseteq T_2\,(G_{k^*})$ and so it suffices to prove that $V_{l,m}\cap T_2\,(G_{k^*})\subseteq V_{l,m}\cap I_k$ in order to get $I_k=T_2\,(G_{k^*})$. Let $f(y_1,\dots,y_l,z_1,\dots,z_m)\in V_{l,m}\cap T_2\,(G_{k^*})$, since $z_1z_2\cdots z_{k+1}$ lies in I_k we can assume that $m\leqslant k$. By Proposition 5, $f(x_1,\dots,x_n)$ is a polynomial identity of E, hence $f(x_1,\dots,x_n)=\sum_{i=1}^h a_i[b_i,c_i,d_i]g_i$ for some polynomials $a_i,\dots,g_i\in F\langle X\rangle$. Since $f(x_1,\dots,x_n)$ is multilinear we can assume that each of these elements is a monomial in $F\langle X\rangle$ and $a_ib_ic_id_ig_i\in V_n$ for all i. So, we have $f(y_1,\dots,y_l,z_1,\dots,z_m)=\psi_{l,m}(f(x_1,\dots,x_n))=\sum_{i=1}^h\psi_{l,m}(a_i[b_i,c_i,d_i]g_i)=\sum_{i=1}^h\bar a_i[\bar b_i,\bar c_i,\bar d_i]\bar g_i$, where $\bar a_i,\dots,\bar g_i$ are monomials in $F\langle Y\cup Z\rangle$ and $\bar a_i\bar b_i\bar c_i\bar d_i\bar g_i\in V_{l,m}$ for all i. Therefore any summand is in the T_2-ideal I_k since it contains the polynomials $[u_1,u_2,u_3]$. This proves the second point of our theorem. Clearly, a similar argument holds in the first case and we are done. $\square$

In order to find the generators for $T_2(G_k)$ and describe its cocharacter and codimension sequences we need some preliminary results involving the notion of *Y-proper polynomials*.

5. Y-proper polynomial identities of G_k, some reductions

Proper polynomials were brought up for the first time by Malcev and Specht in 1950, and constitute a valuable tool when working with polynomial identities of unitary algebras. Drensky in [3] started using them in a quantitative approach to the study of T-ideals of free algebras, developing an idea used by Volichenko [16]. Here we recall briefly some definitions and results.

Let R be any ring, and let $r_1, r_2 \in R$. The *commutator* of r_1 and r_2 is the element $[r_1, r_2]:=r_1r_2 - r_2r_1$. If $k \geqslant 3$ one writes inductively $[r_1, r_2, \ldots, r_k]:=[r_1, \ldots, r_{k-1}]r_k - r_k[r_1, \ldots, r_{k-1}]$, a *left-normed* commutator of *length k*.

When dealing with $\mathbb{Z}_2$-graded algebras the free algebra is $F\langle Y \cup Z\rangle$ and one speaks of *Y-proper polynomials* (see [5, Section 2; 7, Section 2]). These polynomials are the elements of the unitary F-subalgebra B generated by the elements of Z and by all non trivial commutators. Roughly speaking, a polynomial $f \in F\langle Y \cup Z\rangle$ is Y-proper if all the $y \in Y$ occurring in f appear in commutators only. Notice that if $f \in F\langle Z\rangle$ then f is Y-proper.

It is well known (see, for instance, Lemma 1 Section 2 in [7]) that all graded polynomial identities of a superalgebra A follow from the Y-proper ones. This means that the set $T_2(A) \cap B$ generates the whole $T_2(A)$ as a T_2-ideal. Let us denote $B(A):=B/\Big(T_2(A) \cap B\Big)$.

We shall denote $\Gamma_{l,m}$ the set of multilinear polynomials of $V_{l,m}$ which are Y-proper. It is not difficult to see that $\Gamma_{l,m}$ is a left $S_l \times S_m$-submodule of $V_{l,m}$ and the same holds for $\Gamma_{l,m} \cap T_2(A)$. Hence the factor module

$$\Gamma_{l,m}(A):=\frac{\Gamma_{l,m}}{\Gamma_{l,m} \cap T_2(A)}$$

is a $S_l \times S_m$-submodule of $V_{l,m}(A)$. We shall denote by $\xi_{l,m}$ its character (a $\mathbb{Z}_2$*-graded proper cocharacter of* A) and by $\gamma_{l,m}(A)$ its dimension.

The next result relates the structure of $V_{l,m}(A)$ with the structure of $\Gamma_{l,m}(A)$ (see [7] Proposition 1 Section 2). Here and in the following we adopt the notation

$$((l-r) \otimes (\lambda \otimes \mu))^{S_l \times S_m} := ((l-r) \otimes \lambda)^{S_l} \otimes \mu,$$

where $\lambda \vdash r$ and $\mu \vdash m$. With this notation, we have:

Proposition 11. *Let A be a unitary superalgebra, and let $\xi_{r,m}(A)$ be the sequence of proper cocharacters of A. Then*

$$\chi_{l,m}(A) = \sum_{r=0}^{l} ((l-r) \otimes \xi_{r,m}(A))^{S_l \times S_m}.$$

Moreover,

$$c_{l,m}(A) = \sum_{r=0}^{l} \binom{l}{r} \gamma_{r,m}(A).$$

Using Eq. (5) we easily obtain:

Lemma 12. *If $l \equiv 0 \pmod 2$ then $\Gamma_{l,m}$ is generated modulo I by the polynomials*

$$z_{i_1} \ldots z_{i_m}[y_1, y_2] \ldots [y_{l-1}, y_l]$$

That is: for any $f \in \Gamma_{l,m}$ there exists $g \in \Gamma_{0,m}$ such that

$$f(y_1, \ldots, y_l, z_1, \ldots, z_m) \equiv g(z_1, \ldots, z_m)[y_1, y_2] \ldots [y_{l-1}, y_l] \pmod I.$$

In a similar way we obtain:

Lemma 13. *If $l \equiv 1 \pmod 2$ and $m \geqslant 1$ then $\Gamma_{l,m}$ is generated modulo I by the polynomials*

$$z_{i_1} \ldots z_{i_{m-1}}[z_{i_m}, y_1][y_2, y_3] \ldots [y_{l-1}, y_l]$$

That is: for any $f \in \Gamma_{l,m}$ there exists $g \in \Gamma_{1,m}$ such that

$$f(y_1, \ldots, y_l, z_1, \ldots, z_m) \equiv g(z_1, \ldots, z_m, y_1)[y_2, y_3] \ldots [y_{l-1}, y_l] \pmod I.$$

In the even case we also have:

Lemma 14. *Let $l \equiv 0 \pmod 2$ and $f \in \Gamma_{l,m}$.*

(a) *if $l \geqslant k+1$ then $f \in T_2(G_k)$,*
(b) *if $l \leqslant k$ then $f \in T_2(G_k) \Longleftrightarrow g \in T_2(G_{k-l})$,*

In the odd case we obtain:

Lemma 15. *Let $l \equiv 1 \pmod 2$, $m \geqslant 1$ and $f \in \Gamma_{l,m}$.*

(a) *if $l \geqslant k+1$ then $f \in T_2(G_k)$;*
(b) *if $l \leqslant k$ then $f \in T_2(G_k) \Longleftrightarrow g \in T_2(G_{k-l+1})$.*

The above lemmas leads us to study just $\Gamma_{0,m} \cap T_2(G_h)$ and $\Gamma_{1,m} \cap T_2(G_h)$ for all h.

6. The structure of $\boldsymbol{\Gamma_{0,m}}(G_h)$

First we give an upper bound for the proper graded codimension $\gamma_{0,m}(G_h)$. Using the results in Remark 8 and Eq. (5), it is easy to see $\Gamma_{0,m}$ is generated, modulo I, by the polynomials

$$z_{i_1} \ldots z_{i_r} \left[z_{j_1}, z_{j_2}\right] \ldots \left[z_{j_{t-2}}, z_{j_t}\right],$$

where t is even, $r+t=m$ and

- $i_1 < i_2 < \cdots < i_r$;
- $j_1 < j_2 < \cdots < j_t$.

Let $T = \{j_1, \ldots, j_t\} \subseteq \{1, \ldots, m\}$ and let us denote the previous polynomial by $f_T(z_1, \ldots, z_m)$. Therefore any element of $\Gamma_{0,m}$ is, modulo I, a linear combination of polynomials f_T, that is

$$f \equiv \sum_T \alpha_T f_T \pmod I$$

for some $\alpha_T \in F$.

Definition 16. For $m \geqslant 2$ let

$$g_m(z_1, \ldots, z_m) = \sum_{\substack{T \\ |T| \text{ even}}} (-2)^{-\frac{|T|}{2}} f_T,$$

moreover put

$$g_1(z_1) = z_1.$$

By previous definition we obtain:

Lemma 17. *In the free superalgebra $F\langle Y \cup Z\rangle$ the following equivalences hold*:

(a) $$g_{m+1}(z_1,\dots,z_{m+1}) \equiv z_1 g_m(z_2,\dots,z_{m+1}) - \frac{1}{2}[z_1,z_2]\,g_{m-1}(z_3,\dots,z_{m+1}) - \frac{1}{2}\sum_{i=2}^{m-1} z_2z_3\cdots z_i[z_1,z_{i+1}]\,g_{m-i}(z_{i+2},\dots,z_{m+1}) - \frac{1}{2}z_2\cdots z_m[z_1,z_{m+1}] \pmod{I}.$$

(b) $$g_{m+1}(z_1,\dots,z_{m+1}) \equiv g_m(z_1,\dots,z_m)z_{m+1} - \frac{1}{2}g_{m-1}(z_1,\dots,z_{m-1})[z_m,z_{m+1}] - \frac{1}{2}\sum_{i=1}^{m-2} g_i(z_1,\dots,z_i)z_{i+2}\cdots z_m[z_{i+1},z_{m+1}] - \frac{1}{2}z_2\cdots z_m[z_1,z_{m+1}] \pmod{I}.$$

Proposition 18. *The polynomial $g_{h+2}(z_1,\dots,z_{h+2})$ is a $\mathbb{Z}_2$-graded polynomial identity for G_h.*

Proof. Let S be any finite subset of $\mathbb{N}^*$ and let $\varphi_S : F\langle Y\cup Z\rangle \to F\langle Y \cup Z\rangle$ be the endomorphism induced by $\varphi_S(y_i) := y_{|S|+i}$ for all $i \geqslant 1$, and

$$\varphi_S(z_i) := v_i = \begin{cases} y_i & \text{if } i \in S,\\ z_i & \text{if } i \notin S.\end{cases}$$

Notice that, if $T_2(E)$ denotes the T_2-ideal of the natural $\mathbb{Z}_2$-grading on E induced by the map $|\cdot|$, then a multilinear polynomial $f(z_1,\dots,z_m) \in F\langle Z\rangle$ is a graded polynomial identity for the superalgebra G_h if and only if $\varphi_S(f) \in T_2(E)$ for all $S \subseteq \{1,\dots,m\}$ with $|S| \leqslant h$. So we will prove by induction on h that $\varphi_S(g_{h+2}) \in T_2(E)$ for all S such that $|S| \leqslant h$.

Clearly, if $h = 0$ then $\varphi_S(g_{h+2}) = \varphi_\emptyset(g_2) = g_2(z_1,z_2) = z_1z_2 - \frac{1}{2}[z_1,z_2] = \frac{1}{2}(z_1z_2 + z_2z_1)$ and this polynomial belongs to the T_2-ideal $T_2(E)$.

Let us consider $h \geqslant 1$ and fix a subset S of $\{1,\dots,h+2\}$ of cardinality at most h. Although φ_S is not a $\mathbb{Z}_2$-graded endomorphism of the free superalgebra, the T_2-ideal I (given by Definition 9) is invariant under the action of this endomorphism.

Assume that $h+2 \in S$. Since $T_2(E)$ contains the T_2 ideal I and the polynomials $[z_i, y_{h+2}]$, $[y_i, y_{h+2}]$, then by (b) of Lemma 17 we obtain $\varphi_S(g_{h+2}(z_1,\dots,z_{h+2})) \equiv \varphi_S(g_{h+1}(z_1,\dots,z_{h+1}))y_{h+2} \pmod{T_2(E)}$. Let $S' = S \setminus \{h+2\}$, so

$$\varphi_S(g_{h+1}(z_1,\dots,z_{h+1})) = \varphi_{S'}(g_{h+1}(z_1,\dots,z_{h+1})) \in T_2(E)$$

by inductive hypothesis.

Now, we suppose that $h+2 \notin S$. Let k be the maximum in $\{1,\dots,h+1\}\setminus S$, so

$$(v_1,\dots,v_{h+2}) = (v_1,v_2,\dots,v_{k-1},z_k,y_{k+1},\dots,y_{h+1},z_{h+2}).$$

If $i = 0,\dots,k-2$, since the cardinality of the set $S \cap \{i+2,\dots,h+2\}$ is at most $h-i-1$, by inductive hypothesis we obtain $\varphi_S(g_{h+1-i}(z_{i+2},\dots,z_{h+2})) \in T_2(E)$. If $i = k-1$, then we get

$$\begin{aligned}\varphi_S(g_{h+2-k}(z_{k+1},\dots,z_{h+2})) &= g_{h+2-k}(y_{k+1},\dots,y_{h+1},z_{h+2})\\ &\equiv y_{k+1}\cdots y_{h+1}z_{h+2} \pmod{T_2(E)}.\end{aligned}$$

Finally, if $i = k,\dots,h$, then $\varphi_S([z_1,z_{i+1}]) = [v_1,y_{i+1}] \in T_2(E)$. Therefore, by using (a) of Lemma 17 we get

$$\begin{aligned}\varphi_S(g_{h+2}(z_1,\dots,z_{h+2})) \equiv &-\frac{1}{2}v_2\cdots v_{k-1}[v_1,z_k]y_{k+1}\cdots y_{h+1}z_{h+2}\\ &-\frac{1}{2}v_2\cdots v_{k-1}z_ky_{k+1}\cdots y_{h+1}[v_1,z_{h+2}] \pmod{T_2(E)}.\end{aligned}$$

Since $T_2(E)$ contains the polynomials $[y_i,z_j]$ and $z_jz_t + z_tz_j$, we conclude our proof. □

Definition 19. Let J_h be the T_2-ideal of the free superalgebra $F\langle Y \cup Z\rangle$ generated by I and the polynomial $g_{h+2}(z_1,\ldots,z_{h+2})$.

As a consequence of the previous result we obtain $J_h \subseteq T_2(G_h)$, moreover we have:

Lemma 20. *In the free superalgebra $F\langle Y \cup Z\rangle$ we have*:

(a) $z_1 z_2 \cdots z_{h+2} \equiv \sum_{\substack{T\neq\emptyset \\ |T|\text{even}}} -(-2)^{-\frac{|T|}{2}} f_T \pmod{J_h}$
(b) $z_2 \cdots z_{h+2}[z_1, z_{h+3}] \equiv \sum_{T'} \beta_{T'} f_{T'} \pmod{J_h}$

for some $\beta_{T'} \in F$ and $T' \subseteq \{1,\ldots,h+3\}$, moreover if $|T'| = 2$ then $1 \notin T'$.

Proof. The proof of (a) follows immediately by the previous definitions since the polynomial g_{h+2} is in J_h. Therefore, modulo J_h, the monomial $z_1 \cdots z_{h+2}$ is a linear combination of the polynomials f_T corresponding to the non-empty subsets of $\{1,\ldots,h+2\}$ with even cardinality. Hence

$$[z_1 \ldots z_{h+2}, z_{h+3}] \equiv \sum_T \alpha_T [f_T, z_{h+3}] \pmod{J_h}$$

for some $\alpha_T \in F, T \neq \emptyset$. Moreover, if $f_T = z_{a_1} \cdots z_{a_r}[z_{b_1}, z_{b_2}] \cdots [z_{b_{t-1}}, z_{b_t}]$, then, modulo I, we have

$$[f_T, z_{h+3}] \equiv \sum_{s=1}^{r} \beta_s^T z_{a_1} \cdots z_{a_{s-1}} z_{a_{s+1}} \cdots z_{a_r}[z_{a_s}, z_{h+3}][z_{b_1}, z_{b_2}] \cdots [z_{b_{t-1}}, z_{b_t}]$$

and also

$$[z_2 \ldots z_{h+2}, z_{h+3}] \equiv \sum_{s=2}^{h+2} z_2 \ldots z_{s-1} z_{s+1} \ldots z_{h+2}[z_s, z_{h+3}].$$

This implies

$$\begin{aligned} & z_2 \cdots z_{h+2}[z_1, z_{h+3}] \\ & \quad \equiv -\sum_{s=2}^{h+2} z_1 \ldots z_{s-1} z_{s+1} \ldots z_{h+2}[z_s, z_{h+3}] \\ & \qquad + \sum_{\substack{T\neq\emptyset \\ |T|\text{even}}} \alpha_T \sum_{s=1}^{r} \beta_s^T z_{a_1} \cdots z_{a_{s-1}} z_{a_{s+1}} \cdots z_{a_r}[z_{a_s}, z_{h+3}][z_{b_1}, z_{b_2}] \cdots [z_{b_{t-1}}, z_{b_t}]. \end{aligned}$$

By Eq. (5), modulo I one has

$$[z_{a_s}, z_{h+3}][z_{b_1}, z_{b_2}] \cdots [z_{b_{t-1}}, z_{b_t}] \equiv \pm [z_{c_1}, z_{c_2}] \cdots [z_{c_{t+1}}, z_{c_{t+2}}],$$

where $c_1 < \cdots < c_{t+2}$. Since $I \subseteq J_h$ we are done. $\square$

Proposition 21. *For all $m \geqslant 1$, $\Gamma_{0,m}$ is spanned modulo J_h by $\sum_{s=0}^{h}\binom{m-1}{s}$ polynomials.*

Proof. As we said above, $\Gamma_{0,m}$ is spanned modulo I by the polynomials $f_T = z_{i_1} \ldots z_{i_r}[z_{j_1}, z_{j_2}] \ldots [z_{j_{t-1}}, z_{j_t}]$ where $T = \{j_1,\ldots,j_t\}$, $|T| = t$ is even, $r + t = m$, $i_1 < i_2 < \cdots < i_r$ and $j_1 < j_2 < \cdots < j_t$. By using the results of previous lemma, if $m \geqslant h+2$ then, modulo J_h, any polynomial f_T is a linear combination of polynomials f_S where $s := m - |S| \leqslant h+1$. Moreover, if $m \equiv h+1 \pmod 2$ and $s = m - |S| = h+1$ then we can assume that $1 \notin S$. Let us denote by $d_m = d_m(h)$ the number of such polynomials f_S. Using the formula $\binom{m}{s} = \binom{m-1}{s} + \binom{m-1}{s-1}$ we get the following value for d_m:

- If $m \equiv h \pmod 2$ then $d_m = \sum_{\substack{s=0\\ m-s \text{ even}}}^{h+1} \binom{m}{s} = \sum_{s=0}^{h} \binom{m-1}{s}$
- If $m \equiv h+1 \pmod 2$ then $d_m = \binom{m-1}{h} + \sum_{\substack{s=0\\ m-s \text{ even}}}^{h} \binom{m}{s} = \sum_{s=0}^{h} \binom{m-1}{s}$.

Clearly, if $m \leqslant h+1$ we get $\sum_{\substack{s=0\\ m-s \text{ even}}}^{m} \binom{m}{s} = \sum_{s=0}^{m-1} \binom{m-1}{s}$ generators for $\Gamma_{0,m}$ modulo J_h, and we are done because $\binom{m-1}{s} = 0$ for all $s \geqslant m$. $\square$

In order to find the exact value for $\gamma_{0,m}(G_h)$ we examine the structure of the S_m-module $\Gamma_{0,m}(G_h)$. We give

Definition 22. Set $l \leqslant m$, and define

$$p_l(z_1,\ldots,z_m) := \sum_{\sigma} St_l(z_{\sigma(1)}, z_2, \ldots, z_l) z_{\sigma(l+1)} \ldots z_{\sigma(m)},$$

where $St_l(u_1,\ldots,u_l)$ denotes the standard polynomial of degree l.

Remark 23. The polynomial p_l corresponds to the standard Young tableau of the hook partition $(m-l+1, 1^{l-1}) = (1+b, 1^{m-b-1})$. This means that p_l generates an irreducible S_m-module of dimension $\binom{m-1}{l-1} = \binom{m-1}{b}$.

Lemma 24. *If $b := m-l \leqslant h$ then p_l is not a graded identity for G_h.*

Proof. Since $b \leqslant h$ there exist b central elements of odd degree in G_h with pairwise disjoint supports, more precisely we put $a_i = \epsilon_{l+i}\eta_i$ for $i = 1,\ldots,b$.

First we assume that l is even: then $St_l(a_i, \epsilon_2, \ldots, \epsilon_l) = 0$ for all $i \geqslant 1$ while $St_l(\epsilon_1, \epsilon_2, \ldots, \epsilon_l) = l!\epsilon_1 \cdots \epsilon_l$. In this case

$$p_l(\epsilon_1,\ldots,\epsilon_l, a_1,\ldots,a_{m-l}) = l!b!\epsilon_1 \cdots \epsilon_l a_1 \cdots a_{m-l} \neq 0.$$

If l is odd then $St_l(a_i, \epsilon_2, \ldots, \epsilon_l) = a_i St_{l-1}(\epsilon_2,\ldots,\epsilon_l) = (l-1)! a_i \epsilon_2 \cdots \epsilon_l$, for all $i \geqslant 1$. Hence, we obtain

$$\begin{aligned} & p_l(\epsilon_1,\ldots,\epsilon_l, a_1,\ldots,a_{m-l}) \\ &\quad = l!b!\epsilon_1 \cdots \epsilon_l a_1 \cdots a_{m-l} + \sum_{i=1}^{b} (l-1)!b! a_i \epsilon_2 \cdots \epsilon_l \epsilon_1 a_1 \cdots a_{i-1} a_{i+1} \cdots a_{m-l} \\ &\quad = (l+b)(l-1)!b!\epsilon_1 \cdots \epsilon_l a_1 \cdots a_{m-l} \neq 0. \quad \square \end{aligned}$$

In the following result we summarize our results about the structure of $\Gamma_{0,m}(G_h)$.

Proposition 25. *Let $m \geqslant 1$ then*

(a) $\Gamma_{0,m}(G_h) = \Gamma_{0,m}(J_h)$
(b) $\gamma_{0,m}(G_h) = \sum_{s=0}^{h} \binom{m-1}{s}$
(c) $\xi_{0,m}(G_h) = \sum_{s=0}^{h} \emptyset \otimes \mu_s$,

where $\mu_s = \left(1+s, 1^{m-s-1}\right) \vdash m$.

Proof. Since $J_h \subseteq T_2(G_h)$ we obtain by Proposition 21 $\gamma_{0,m}(G_h) \leqslant d_m(h) = \sum_{s=0}^{h} \binom{m-1}{s}$. On the other hand, the irreducible module generated by the polynomial p_l determines an irreducible component of $\Gamma_{0,m}(G_h)$ of dimension $\binom{m-1}{l-1} = \binom{m-1}{b}$ for all l such that $m - l = b \leqslant h$. Since these irreducible components are all distinct we obtain $\gamma_{0,m}(G_h) \geqslant \sum_{b=0}^{h} \binom{m-1}{b}$ and we are done. $\square$

More generally, in the even case we obtain

Corollary 26. *Let* $l \equiv 0 \pmod 2$ *and* $m \geqslant 1$, *if* $l \leqslant k$ *then*:

(a) $\gamma_{l,m}(G_k) = \sum_{s=0}^{k-l} \binom{m-1}{s}$
(b) $\xi_{l,m}(G_k) = \sum_{s=0}^{k-l} \lambda_l \otimes \mu_s$,

where $\lambda_l = (1^l) \vdash l$ *and* $\mu_s = (s+1, 1^{m-1-s}) \vdash m$.

Proof. By Lemma 14 we have a linear isomorphism between $\Gamma_{l,m}(G_k)$ and $\Gamma_{0,m}(G_{k-l})$ and this proves the first statement. The second one follows from Eq. (5) and the decomposition given in Proposition 25. $\square$

We finish this section by studying the exceptional case when l is even and $m = 0$. We have:

Proposition 27. *Let* $l \equiv 0 \pmod 2$, *if* $l \leqslant k$ *then*:

(a) $\gamma_{l,0}(G_k) = 1$;
(b) $\xi_{l,0}(G_k) = \lambda_l \otimes \emptyset$,

where $\lambda_l = (1^l) \vdash l$.

7. The structure of $\Gamma_{1,m}(G_k)$

As in the even case we obtain that $\Gamma_{1,m}$ is spanned modulo I by the following polynomials:

$$f'_T(z_1,\ldots,z_m,y) = z_{i_1}\cdots z_{i_r}[z_{j_1},z_{j_2}]\cdots[z_{j_{t-2}},z_{j_{t-1}}][z_{j_t},y],$$

where $T = \{j_1,\ldots,j_t\} \subseteq \{1,\ldots,m\}$, $|T| = t$ is odd, $r + t = m$ and

- $i_1 < i_2 < \cdots < i_r$;
- $j_1 < j_2 < \cdots < j_t$.

In this case G_h satisfies the following polynomial identities:

- $[g_{h+1}(z_1,\ldots,z_{h+1}),y]$,
- $g_{h+1}(z_1,\ldots,z_{h+1})[z_{h+2},y]$.

As a consequence of these graded polynomial identities of G_h, by using similar arguments to those given in the proofs of Lemma 20 and Proposition 21, we obtain:

Proposition 28. *Let* J'_h *be the* T_2*-ideal generated by* I *and the polynomials* $[g_{h+1}(z_1,\ldots,z_{h+1}),y]$, $g_{h+1}(z_1,\ldots,z_{h+1})[z_{h+2},y]$, *then, for all* $m \geqslant 1$, $\Gamma_{1,m}$ *is spanned modulo* J'_h *by* $\sum_{s=0}^{h-1}\binom{m-1}{s}$ *polynomials. More precisely the generators of* $\Gamma_{1,m}(J'_h)$ *are the polynomials*

$$f'_T = z_{i_1}\cdots z_{i_r}[z_{j_1},z_{j_2}]\cdots[z_{j_{t-2}},z_{j_{t-1}}][z_{j_t},y],$$

where $i_1 < i_2 < \ldots < i_r$ *and* $j_1 < j_2 < \ldots < j_t$, $T = \{j_1, \ldots, j_t\}$ *runs over all the subsets of* $\{1, \ldots, m\}$ *such that* $|T| = t$ *is odd*, $r = m - t \leqslant h$, *and* $1 \notin T$ *if* $r = h$.

Let us consider the following polynomials in $\Gamma_{1,m}$:

Definition 29. Set $l \leqslant m$, and define

- $q_l(y, z_1, \ldots, z_l) := \sum_\sigma (-1)^\sigma [y, z_{\sigma(1)}] z_{\sigma(2)} \cdots z_{\sigma(l)}$
- $p'_l(y, z_1, \ldots, z_m) := \sum_\sigma q_l(y, z_{\sigma(1)}, z_2, \ldots, z_l) z_{\sigma(l+1)} \ldots z_{\sigma(m)}$.

Remark 30. The polynomial p'_l corresponds to the pair of standard Young tableaux (λ_1, μ_{m-l}) where $\lambda_1 = (1) \vdash 1$ and μ_{m-l} is the hook partition $(m - l + 1, 1^{l-1})$ of m. This means that p'_l generates an irreducible $S_1 \times S_m$-module of dimension $\binom{m-1}{l-1}$.

We have:

Lemma 31. *If* $b := m - l \leqslant h - 1$ *then* p'_l *is not a graded identity for* G_h.

Proof. Since $m - l = b \leqslant h - 1$ there exist b central elements $a_1, \ldots, a_b$ of odd degree in G_h with pairwise disjoint support and such that $\eta = \eta_h \notin supp(a_i)$ for all $i \geqslant 1$. More precisely we put $a_i = \epsilon_{l+i}\eta_i$ for $i = 1, \ldots, b$.

First we assume that l is odd: then $q_l(\eta, a_i, \epsilon_2, \ldots, \epsilon_l) = 0$ for all $i \geqslant 1$ while $q_l(\eta, \epsilon_1, \epsilon_2, \ldots, \epsilon_l) = 2 \cdot l!\eta\epsilon_1 \cdots \epsilon_l$. In this case

$$p'_l(\eta, \epsilon_1, \ldots, \epsilon_l, a_1, \ldots, a_{m-l}) = 2 \cdot l!b!\eta\epsilon_1 \cdots \epsilon_l a_1 \cdots a_{m-l} \neq 0.$$

If l is even then $q_l(\eta, a_i, \epsilon_2, \ldots, \epsilon_l) = -2 \cdot (l-1)!\eta a_i \epsilon_2 \cdots \epsilon_l$, for all $i = 1, \ldots, b$. Hence, we obtain

$$\begin{aligned} &p'_l(\eta, \epsilon_1, \ldots, \epsilon_l, a_1, \ldots, a_{m-l}) \\ &\quad = 2 \cdot l!b!\eta\epsilon_1 \cdots \epsilon_l a_1 \cdots a_{m-l} - \sum_{i=1}^{b} 2 \cdot (l-1)!b!\eta a_i \epsilon_2 \cdots \epsilon_l \epsilon_1 a_1 \cdots a_{i-1} a_{i+1} \cdots a_{m-l} \\ &\quad = 2 \cdot (l+b)(l-1)!b!\eta\epsilon_1 \cdots \epsilon_l a_1 \cdots a_{m-l} \neq 0. \quad \square \end{aligned}$$

As a consequence we obtain the description of $\Gamma_{1,m}(G_h)$.

Proposition 32. *Let* $m \geqslant 1$ *then*

(a) $\Gamma_{1,m}(G_h) = \Gamma_{1,m}(J'_h)$.
(b) $\gamma_{1,m}(G_h) = \sum_{s=0}^{h-1} \binom{m-1}{s}$.
(c) $\xi_{1,m}(G_h) = \sum_{s=0}^{h-1} \lambda_1 \otimes \mu_s$,

where $\lambda_1 = (1) \vdash 1$ *and* $\mu_s = (1 + s, 1^{m-s-1}) \vdash m$.

Proof. Since $J'_h \subseteq T_2(G_h)$ we obtain by Proposition 28 $\gamma_{1,m}(G_h) \leqslant \sum_{s=0}^{h-1} \binom{m-1}{s}$. On the other hand, the irreducible module generated by the polynomial p'_l determines an irreducible component of $\Gamma_{1,m}(G_h)$ of dimension $\binom{m-1}{l-1} = \binom{m-1}{b}$ for all l such that $b = m - l \leqslant h - 1$. Since these irreducible components are all distinct we obtain $\gamma_{1,m}(G_h) \geqslant \sum_{b=0}^{h-1} \binom{m-1}{b}$ and we are done. $\square$

As in the even case we also obtain:

Corollary 33. *Let* $l \equiv 1 \pmod 2$ *and* $m \geqslant 1$, *if* $l \leqslant k$ *then*:

(a) $\gamma_{l,m}(G_k) = \sum_{s=0}^{k-l} \binom{m-1}{s}$
(b) $\xi_{l,m}(G_k) = \sum_{s=0}^{k-l} \lambda_l \otimes \mu_s$,

where $\lambda_l = (1^l) \vdash l$ *and* $\mu_s = \left(s+1, 1^{m-1-s}\right) \vdash m$.

Proof. By Lemma 15 we have a linear isomorphism between $\Gamma_{l,m}(G_k)$ and $\Gamma_{1,m}(G_{k-l+1})$ and this proves the first statement. The second one follows from the Eq. (5) and the decomposition given in Proposition 32. □

Remark 34. Clearly, if l is odd and $m = 0$ then $\Gamma_{l,0}(G_k) = 0$ because, by Remark 8, any Y-proper polynomial of odd degree in $F\langle Y\rangle$ is a graded polynomial identity of G_k.

8. The main result on $T_2(G_k)$

Let us recall that $(G_k, \|\cdot\|)$ denotes the pair $(E, \|\cdot\|_k)$, where E is the Grassmann algebra generated by an infinite dimensional vector space L over the field F of characteristic zero and $\|\cdot\|_k$ induces on E the $\mathbb{Z}_2$-grading defined in Eqs. (1) and (2).

We set

$$e(k) = \begin{cases} k & \text{if } k \text{ is even} \\ k-1 & \text{if } k \text{ is odd} \end{cases}$$

and we obtain the following decomposition for the graded cocharacter sequences of G_k.

Proposition 35. *Let* $\lambda_a = (l-a, 1^a) \vdash l$ *and* $\mu_b = \left(1+b, 1^{m-b-1}\right) \vdash m$ *be the hook partitions of* l *and* m *with leg* a *and arm* b *respectively. The graded cocharacter sequences of the superalgebra* $(G_k, \|\cdot\|)$ *are the following*:

- $\chi_{l,0}(G_k) = \sum_{a=0}^{l-1} \lambda_a \otimes \emptyset \ (l \leqslant k)$
- $\chi_{l,0}(G_k) = \sum_{a=0}^{e(k)} \lambda_a \otimes \emptyset \ (l \geqslant k+1)$
- $\chi_{0,m}(G_k) = \sum_{b=0}^{m-1} \emptyset \otimes \mu_b \ (m \leqslant k)$
- $\chi_{0,m}(G_k) = \sum_{b=0}^{k} \emptyset \otimes \mu_b \ (m \geqslant k+1)$
- $\chi_{l,m}(G_k) = \sum_{a=0}^{l-1} \sum_{b=0}^{m-1} m_{a,b}\, (\lambda_a \otimes \mu_b) \ \ (l, m \geqslant 1)$
  where the multiplicities $m_{a,b}$ *of the irreducible* $S_l \times S_m$*-character* $\lambda_a \otimes \mu_b$ *satisfy the following equalities*:

(i) $m_{a,b} = 2$, *if* $a + b \leqslant k-1$.
(ii) $m_{a,b} = 1$, *if* $a + b = k$.
(iii) $m_{a,b} = 0$, *otherwise*.

Proof. Let ϱ_r be the S_r-irreducible character associated to the partition (1^r) of r, then by the Young-rule we obtain $((l-r) \otimes \varrho_r)^{S_l} = \lambda_{r-1} + \lambda_r$, for $1 \leqslant r \leqslant l-1$. Therefore the first and the second point follows from Propositions 11, 27, Lemma 14 and Remark 34. Clearly, $\chi_{0,m}(G_k) = \xi_{0,m}(G_k)$ and so we obtain the other two cases from Proposition 25. Finally, we consider the case when $l, m \geqslant 1$. Then, as above, $\chi_{l,m}(G_k) = \sum_{r=0}^{l} \left((l-r) \otimes \xi_{r,m}(G_k)\right)^{S_l \times S_m}$. Using the Young-rule and looking to the decompositions of $\xi_{r,m}(G_k)$ obtained in the previous sections, we have that the irreducible components of $\chi_{l,m}(G_k)$ are of type $\lambda_a \otimes \mu_b$ for some $a, b \geqslant 0$. Clearly the multiplicity $m_{a,b}$ of $\lambda_a \otimes \mu_b$ is non-zero if and only if either the irreducible character $\varrho_a \otimes \mu_b$ appears in the decomposition of $\xi_{a,m}(G_k)$ or $\varrho_{a+1} \otimes \mu_b$ occurs in the decomposition of $\xi_{a+1,m}(G_k)$.

At the light of Corollaries 26, 33, this provides the values of the multiplicities as stated. $\square$

About codimension sequences of G_k, we start with a preliminary result about sums of certain binomial coefficients.

Lemma 36. *Let $l, k, q \geqslant 0$ then*

$$\sum_{r=0}^{k} \binom{l}{r} \sum_{s=0}^{k-r} \binom{q}{s} = \sum_{t=0}^{k} \binom{l+q}{t}.$$

Proof. First, we notice that if $k > n = l + q$ then

$$\sum_{t=0}^{k} \binom{l+q}{t} = \sum_{t=0}^{n} \binom{l+q}{t} = 2^n$$

and

$$\sum_{r=0}^{k} \binom{l}{r} \sum_{s=0}^{k-r} \binom{q}{s} = \sum_{r=0}^{l} \binom{l}{r} \sum_{s=0}^{q} \binom{q}{s} = 2^l 2^q = 2^n,$$

since $k > l$ and for each $r = 0, \ldots, l$ one has $k - r \geqslant k - l > q$.

Therefore the result follows by induction on n over all triple (k, l, q) such that $k \leqslant n = l + q$. $\square$

The main result about codimensions of G_k is:

Proposition 37. *The graded codimension sequences of the superalgebra $(G_k, \|\cdot\|)$ are the following*:

- $c_{n-m,m}(G_k) = \sum_{t=0}^{k} \binom{n-1}{t}$ $(m \geqslant 1)$
- $c_{n,0}(G_k) = \sum_{t=0}^{e(k)} \binom{n-1}{t}$
- $c_n^{\mathbb{Z}_2}(G_k) = \begin{cases} 2^n \sum_{t=0}^{k} \binom{n-1}{t} & \text{if } k \text{ is even,} \\ 2^n \sum_{t=0}^{k} \binom{n-1}{t} - \binom{n-1}{k} & \text{if } k \text{ is odd.} \end{cases}$

Proof. By Proposition 11 we have $c_{n-m,m}(G_k) = \sum_{r=0}^{n-m} \binom{n-m}{r} \gamma_{r,m}(G_k)$. Notice that $\gamma_{r,m}(G_k) = 0$ if $r \geqslant k+1$, hence $c_{n-m,m}(G_k) = \sum_{r=0}^{k} \binom{n-m}{r} \gamma_{r,m}(G_k)$. Therefore, if $m \geqslant 1$, the result follows from Corollaries 26, 33 and Lemma 36.

When $m = 0$, then $\gamma_{r,0}(G_k) = 0$ for all odd values of r. Thus by Proposition 27 we obtain $c_{n,0}(G_k) = \sum_{r=0}^{k} \binom{n}{r} \gamma_{r,0}(G_k) = \sum_{\substack{r=0 \\ r \text{ even}}}^{k} \binom{n}{r}$ and this proves the second point of our statement. Finally, we have $c_n^{\mathbb{Z}_2}(G_k) = \sum_{m=0}^{n} \binom{n}{m} c_{n-m,m}(G_k)$ and by previous steps we are done. $\square$

Finally, we can prove the main result of this section:

Theorem 38. *Let $T_2(G_k)$ be the T_2-ideal of the graded polynomial identities for the superalgebra $(G_k, \|\cdot\|)$. Put $U = Y \cup Z$, then $T_2(G_k)$ is generated by the set of the following polynomials*:

(a) $[u_1, u_2, u_3]$;
(b) $[y_1, y_2] \cdots [y_{k-1}, y_k][y_{k+1}, u_{k+2}]$ *if k is even;*
(c) $[y_1, y_2] \cdots [y_{k-2}, y_{k-1}][y_k, y_{k+1}]$ *if k is odd;*
(d) $g_{k-l+2}(z_1, \ldots, z_{k-l+2})[y_1, y_2] \cdots [y_{l-1}, y_l]$ $(\forall\, l \leqslant k,\ l\ \textit{even})$;
(e) $[g_{k-l+2}(z_1, \ldots, z_{k-l+2}), y_1][y_2, y_3] \cdots [y_{l-1}, y_l]$ $(\forall\, l \leqslant k,\ l\ \textit{odd})$;
(f) $g_{k-l+2}(z_1, \ldots, z_{k-l+2})[z_{k-l+3}, y_1][y_2, y_3] \cdots [y_{l-1}, y_l]$ $(\forall\, l \leqslant k,\ l\ \textit{odd})$.

Proof. Let $\mathcal{P}_k$ be the T_2-ideal of $F\langle Y \cup Z\rangle$ generated by the polynomials listed above accordingly to the parity of k. Moreover, let I, J_h, J'_h be the T_2-ideals defined in the previous sections $(h \in \mathbb{N})$. Clearly $I \subseteq \mathcal{P}_k$ and it is easy to verify that $\mathcal{P}_k \subseteq T_2(G_k)$. Hence it is enough to show that $\Gamma_{l,m} \cap T_2(G_k) \subseteq \Gamma_{l,m} \cap \mathcal{P}_k$. Assume first that k and l are both even and let $f(y_1, \ldots, y_l, z_1, \ldots, z_m)$ be a Y-proper graded polynomial identity of G_k. Since l is even then, by Lemma 12, there exists $g(z_1, \ldots, z_m) \in \Gamma_{0,m}$ such that $f \equiv g[y_1, y_2] \cdots [y_{l-1}, y_l]$ (mod I). Since $I \subseteq \mathcal{P}_k$ the same conclusion holds (mod $\mathcal{P}_k$). Since $[y_1, y_2] \cdots [y_{k-1}, y_k][y_{k+1}, u_{k+2}] \in \mathcal{P}_k$ we can assume that $l \leqslant k$ and so by Lemma 14 we obtain that $g(z_1, \ldots, z_m)$ is a graded polynomial identity of the superalgebra G_{k-l}. Put $h = k - l$, then by Proposition 25 we have that g belongs to the T_2-ideal J_h. Hence, there exist certain monomials $v_i, w_i \in F\langle Y \cup Z\rangle$ and some graded endomorphisms φ_i of the free superalgebra such that $g \equiv \sum_i v_i \varphi_i(g_{h+2}) w_i$ (mod I). Since the commutators $[u_1, u_2]$ are central elements in the superalgebra $\frac{F\langle Y \cup Z\rangle}{I}$ we obtain:

$$f \equiv g[y_1, y_2] \cdots [y_{l-1}, y_l] \equiv \sum_i v_i \varphi_i(g_{h+2})[y_1, y_2] \cdots [y_{l-1}, y_l] w_i \ (\mathrm{mod}\ I).$$

Since $g_{h+2}(z_1, \ldots, z_{h+2})$ and $[y_1, y_2] \cdots [y_{l-1}, y_l]$ are polynomials in disjoint sets of indeterminates we can assume that $\varphi_i(y_j) = y_j$ for all i, j. This implies that each summand in the previous sum is an element of the T_2-ideal generated by the polynomial $g_{k-l+2}(z_1, \ldots, z_{k-l+2})[y_1, y_2] \cdots [y_{l-1}, y_l]$ which is one of the generators of $\mathcal{P}_k$. Since $I \subseteq \mathcal{P}_k$ we have the desired conclusion $f \in \mathcal{P}_k$. Similar arguments hold for the remaining cases and we are done. $\square$

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